CS380: Computer Graphics
3D Transformation

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Course URL:
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Class Objectives

● Understand the diff. between points and vectors
● Understand the frame
● Represent transformations in local and global frames
A Question?

- Suppose you have 2 frames and you know the coordinates of a point relative in one frame
  - How would you compute the coordinate of your point relative to the other frame?
  - (Generalized question to the mapping problem that we went over in the class)
Revisit: Mapping from World to Screen

Viewable world

World

NDC

Screen

Window

$x_w$

$x_n$

$x_s$
Geometry

- A part of mathematics concerned with questions of size, shape, and relative positions of figures

- Coordinates are used to represent points and vectors
  - We will learn that they are just a naming scheme
  - The same point can be described by different coordinates
  - Both vectors and points expressed by coordinates, but they are very different

KAIST
(50, 160)
Go 7 miles southwest
Scalar Fields

- A scalar field \( S \) is a set on which addition \((+))\) and multiplication \((\cdot))\) are defined and following conditions hold:
  - \( S \) is closed for addition and multiplication
    \[ \forall a, b \in S \quad a + b \in S \quad a \cdot b \in S \]
  - These operators commute, associate, and distribute
    \[ \forall a, b, c \in S \]
    \[ a + b = b + a \quad a \cdot b = b \cdot a \]
    \[ a + (b + c) = (a + b) + c \quad a \cdot (b \cdot c) = (a \cdot b) \cdot c \]
    \[ a \cdot (b + c) = a \cdot b + a \cdot c \]
Scalar Fields – cont’d

- A scalar field $\mathbf{S}$ is a set on which addition ($+$) and multiplication ($\cdot$) are defined and following conditions hold:
  - Both operators have a unique identity element
    \[ a + 0 = a, \quad a \cdot 1 = a \]
  - Each element has a unique inverse under both operators
    \[ a + (-a) = 0, \quad a \cdot a^{-1} = 1 \]
Examples of Scalar Fields

- Real numbers
- Complex numbers (given the standard definitions for addition and multiplication)
- Rational numbers
- Notation: we will represent scalars by lower case letters
  \[ a, b, c, \ldots \] are scalar variables
**Vector Spaces**

- **A vector (or linear) space** \( V \) over a scalar field \( S \) consists of a set on which the following two operators are defined and the following conditions hold:

- **Two operators for vectors:**
  - Vector-vector addition
    \[ \forall \vec{u}, \vec{v} \in V \quad \vec{u} + \vec{v} \in V \]
  - Scalar-vector multiplication
    \[ \forall \vec{u} \in V, \forall a \in S \quad a\vec{u} \in V \]

- **Notation:**
  - **Vector**
    \[ \vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a & b & c \end{bmatrix}^t \]
Vector Spaces

- Vector-vector addition
  - Commutes and associates
    \[ \vec{u} + \vec{v} = \vec{v} + \vec{u} \quad \vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w} \]

- An additive identity and an additive inverse for each vector
  \[ \vec{u} + \vec{0} = \vec{u} \quad \vec{u} + (\vec{-u}) = \vec{0} \]

- Scalar-vector multiplication distributes
  \[ (a + b)\vec{u} = a\vec{u} + b\vec{u} \quad a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v} \]
Example Vector Spaces

- Geometric vectors (directed segments)
  \[ \mathbf{u}, \mathbf{v}, \mathbf{w} \]
  \[ \mathbf{u} + \mathbf{v} = \mathbf{w} \]
  \[ 2\mathbf{u} \]

- N-tuples of scalars
  \[ \mathbf{u} = (1,3,7)^t \quad \mathbf{u} + \mathbf{v} = (3,5,4)^t = \mathbf{w} \]
  \[ \mathbf{v} = (2,2,-3)^t \quad 2\mathbf{u} = (2,6,14)^t \]
  \[ \mathbf{w} = (3,5,4)^t \quad -\mathbf{v} = (-2,-2,3)^t \]

- We can use N-tuples to represent vectors
Basis Vectors

- A **vector basis** is a subset of vectors from $V$ that can be used to generate any other element in $V$, using just additions and scalar multiplications.

- A basis set, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, is **linearly dependent** if:

  $$\exists a_1, a_2, \ldots, a_n \neq 0 \text{ such that } \sum_{i=0}^{n} a_i \mathbf{v}_i = 0$$

- Otherwise, the basis set is **linearly independent**.
  - A linearly independent basis set with $i$ elements is said to **span** an $i$-dimensional vector space.
Vector Coordinates

- A linearly independent basis set can be used to uniquely name or address a vector
  - This is done by assigning the vector coordinates as follows:
    \[ x = \sum_{i=1}^{3} c_i \vec{v}_i = [\vec{V}_1 \quad \vec{V}_2 \quad \vec{V}_3] \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \vec{V}^t \mathbf{c} \]

- Note: we’ll use bold letters to indicate tuples of scalars that are interpreted as coordinates

- Our vectors are still abstract entities
  - So how do we interpret the equation above?
Interpreting Vector Coordinates

Valid Interpretation

Equally Valid Interpretation

Remember, vectors don’t have any notion of position
Points

- Conceptually, points and vectors are very different
  - A point \( \mathbf{p} \) is a place in space
  - A vector \( \mathbf{v} \) describes a direction independent of position (pay attentions notations)
How Vectors and Points Differ

- The operations of addition and multiplication by a scalar are well defined for vectors
  - Addition of 2 vectors expresses the concatenation of 2 “motions”
  - Multiplying a vector by some factor scales the motion
- These operations does not make sense for points
Making Sense of Points

- Some operations **do make sense** for points
  - Compute a vector that describes the motion from one point to another:
    \[ \mathbf{p} - \mathbf{q} = \mathbf{v} \]
  
  - Find a new point that is some vector away from a given point:
    \[ \mathbf{q} + \mathbf{v} = \mathbf{p} \]
A Basis for Points

- Key distinction between vectors and points: points are *absolute*, vectors are *relative*
- Vector space is completely defined by a set of basis vectors
- The space that points live in requires the specification of an absolute origin

\[ \mathbf{p} = \mathbf{o} + \sum_i \mathbf{v}_i \mathbf{c}_i = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix} \]

Notice how 4 scalars (one of which is 1) are required to identify a 3D point
Frames

- Points live in **Affine spaces**
- Affine-basis-sets are called **frames**

\[ f^t = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad 0] \]

- Frames can describe vectors as well as points

\[
\begin{bmatrix}
\mathbf{v}_1 \\
\mathbf{v}_2 \\
\mathbf{v}_3 \\
\mathbf{0}
\end{bmatrix}
= 
\begin{bmatrix}
c_1 \\
c_2 \\
c_3 \\
1
\end{bmatrix}
\]

\[
\begin{bmatrix}
\mathbf{v}_1 \\
\mathbf{v}_2 \\
\mathbf{v}_3 \\
\mathbf{0}
\end{bmatrix}
= 
\begin{bmatrix}
c_1 \\
c_2 \\
c_3 \\
0
\end{bmatrix}
\]
Pictures of Frames

- Graphically, we will distinguish between vector bases and affine bases (frames) using the following convention.
A Consistent Model

- Behavior of affine frame coordinates is completely consistent with our intuition
  - Subtracting two points yields a vector
  - Adding a vector to a point produces a point
  - If you multiply a vector by a scalar you still get a vector
  - Scaling points gives a nonsense 4th coordinate element in most cases

\[
\begin{bmatrix}
  a_1 \\
  a_2 \\
  a_3 \\
  1
\end{bmatrix}
- \begin{bmatrix}
  b_1 \\
  b_2 \\
  b_3 \\
  1
\end{bmatrix}
= \begin{bmatrix}
  a_1 - b_1 \\
  a_2 - b_2 \\
  a_3 - b_3 \\
  0
\end{bmatrix}
\]

\[
\begin{bmatrix}
  a_1 \\
  a_2 \\
  a_3 \\
  1
\end{bmatrix}
+ \begin{bmatrix}
  v_1 \\
  v_2 \\
  v_3 \\
  0
\end{bmatrix}
= \begin{bmatrix}
  a_1 + v_1 \\
  a_2 + v_2 \\
  a_3 + v_3 \\
  1
\end{bmatrix}
\]
Homogeneous Coordinates

- Notice why we introduce **homogeneous coordinates**, based on simple logical arguments
  - Remember that coordinates are not geometric; they are just scales for basis elements
  - Thus, you should not be bothered by the fact that our coordinates suddenly have 4 numbers

- 3D homogeneous coordinates refer to an affine frame with its 3 basis vectors and origin point
  - 4 coordinates make sense in this aspect
  - 4th coordinate can have one of two values, [0,1], indicating if whether the coordinates name a vector or a point
Affine Combinations

There are certain situations where it makes sense to scale and add points

- Suppose you have two points, one scaled by $\alpha_1$ and the other scaled by $\alpha_2$
- If we restrict the sum of these alphas, $\alpha_1 + \alpha_2 = 1$, we can assure that the result will have $1$ as it’s 4th coordinate value

$$
\begin{bmatrix}
\alpha_1 & a_1 \\
\alpha_2 & a_2 \\
\alpha_3 & a_3 \\
1 & 1
\end{bmatrix} +
\begin{bmatrix}
\beta_1 & b_1 \\
\beta_2 & b_2 \\
\beta_3 & b_3 \\
1 & 1
\end{bmatrix} =
\begin{bmatrix}
\alpha_1 a_1 + \alpha_2 b_1 \\
\alpha_1 a_2 + \alpha_2 b_2 \\
\alpha_1 a_3 + \alpha_2 b_3 \\
\alpha_1 + \alpha_2
\end{bmatrix} =
\begin{bmatrix}
\alpha_1 a_1 + \alpha_2 b_1 \\
\alpha_1 a_2 + \alpha_2 b_2 \\
\alpha_1 a_3 + \alpha_2 b_3 \\
1
\end{bmatrix}
$$

But, is it a point?
Affine Combinations

• Can be thought of as a constrained-scaled addition
  • Defines all points that share the line connecting our two initial points

• Can be extended to 3, 4, or any number of points (e.g., barycentric coordinates)
Affine Transformations

- We can apply transformations to points using a matrix.
  - Need to use 4 by 4 matrices since our basis set has four components.
  - Also, limit ourselves to transforms that preserve the integrity of our points and vectors; point to point, vector to vector.

\[
\begin{bmatrix}
\tilde{v}_1 & \tilde{v}_2 & \tilde{v}_3 & \tilde{o}
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3 \\
1
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
\tilde{v}_1 & \tilde{v}_2 & \tilde{v}_3 & \tilde{o}
\end{bmatrix}
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34} \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3 \\
1
\end{bmatrix}
\]

- This subset of matrices is called the affine subset.
An Example
Composing Transformations

- Represent a series of transformations
  - E.g., want to translate with $T$ and, then, rotate with $R$

- Then, the series is represented by:

$$\dot{p} = \dot{w}^t c \Rightarrow \dot{p}' = \dot{w}^t R T c = \dot{w}^t (R(Tc)) = \dot{w}^t (Rc') = \dot{w}^t c''$$

- Each step in the process can be considered as a change of coordinates

- Alternatively, we could have considered the same sequence of operations as:

$$\dot{p} = \dot{w}^t c \Rightarrow \dot{p}' = \dot{w}^t R T c = ((\dot{w}^t R)T)c = (\dot{m}^t T)c = \dot{e}^t c$$

  where each step is considered as a change of basis
An Example

- These are alternate interpretations of the same transformations
  - The left and right sequence are considered as a transformation about a *global frame and local frames*
Same Point in Different Frames

- Suppose you have 2 frames and you know the coordinates of a point relative in one frame
  - How would you compute the coordinate of your point relative to the other frame?
    \[ \mathbf{p} = w^t \mathbf{c} = z^t \]
  - Suppose that my two frames are related by the transform \( S \) as shown below:
    \[ z^t = w^t S \quad \text{and} \quad w^t = z^t S^{-1} \]
  - Then, the coordinate for the point in second frame is simply:
    \[ \mathbf{p} = w^t \mathbf{c} = z^t S^{-1} \mathbf{c} = z^t (S^{-1} \mathbf{c}) = z^t \mathbf{d} \]

Substitute for the frame
Reorganize & reinterpret
Revisit: Mapping from World to Screen

World $x_w$  

Viewable world  

NDC $x_n$  

Screen $x_s$  

Window
Class Objectives were:

- Understand the diff. between points and vectors
- Understand the frame
- Represent transformations in local and global frames
Quiz Assignment
Next Time

- Modeling and viewing transformations
Homework

● Go over the next lecture slides before the class
● Watch 2 SIGGRAPH videos and submit your summaries before every Tue. class
Any Questions?

● Come up with one question on what we have discussed in the class and submit at the end of the class
  ● 1 for already answered questions
  ● 2 for typical questions
  ● 3 for questions with thoughts or that surprised me

● Submit at least four times during the whole semester