CS482: Monte Carol Integration

Sung-Eui Yoon
(윤성의)

http://sglab.kaist.ac.kr/~sungeui/ICG

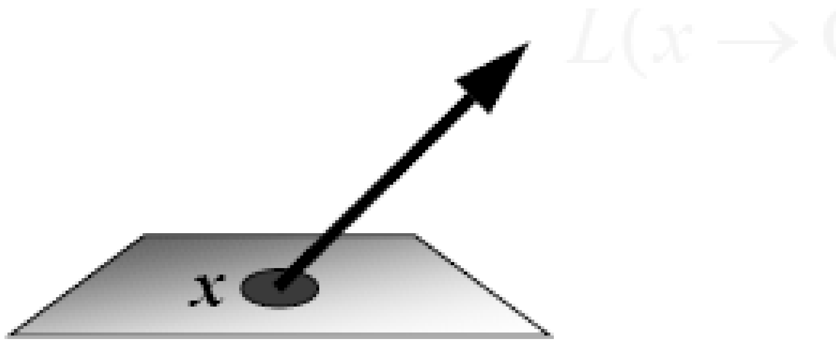
KAIST

Class Objectives

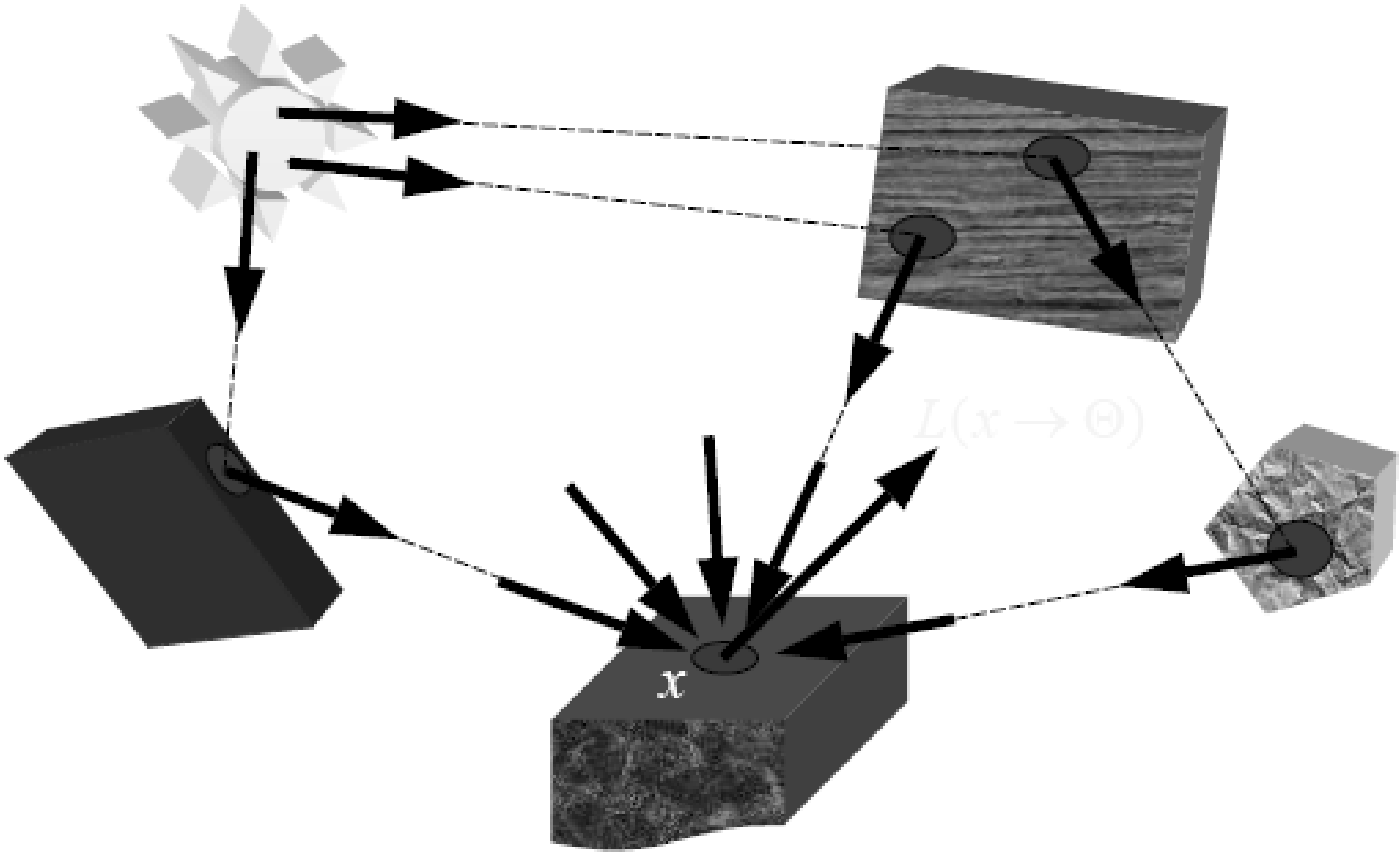
- Sampling approach for solving the rendering equation
  - Monte Carlo integration
  - Estimator and its variance

Radiance Evaluation

- **Fundamental problem in GI algorithm**
  - **Evaluate radiance at a given surface point in a given direction**
  - **Invariance defines radiance everywhere else**



Radiance Evaluation

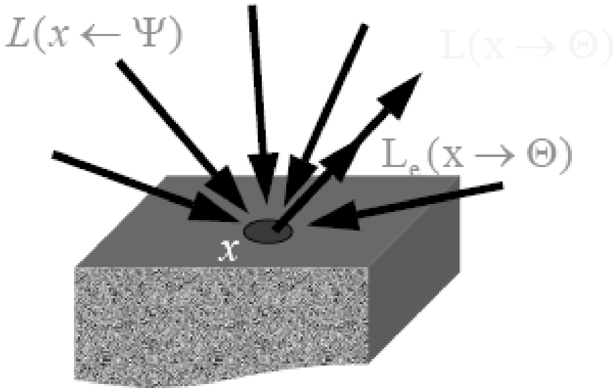


… find paths between sources and surfaces to be shaded

Why Monte Carlo?

- Radiace is hard to evaluate

$$L(x \rightarrow \Theta) = L_e(x \rightarrow \Theta) + \int_{\Omega_x} f_r(\Psi \leftrightarrow \Theta) \cdot L(x \leftarrow \Psi) \cdot \cos(\Psi, n_x) \cdot d\omega_\Psi$$



From kavita's slides

- Sample many paths
  - Integrate over all incoming directions
- Analytical integration is difficult
  - Need numerical techniques

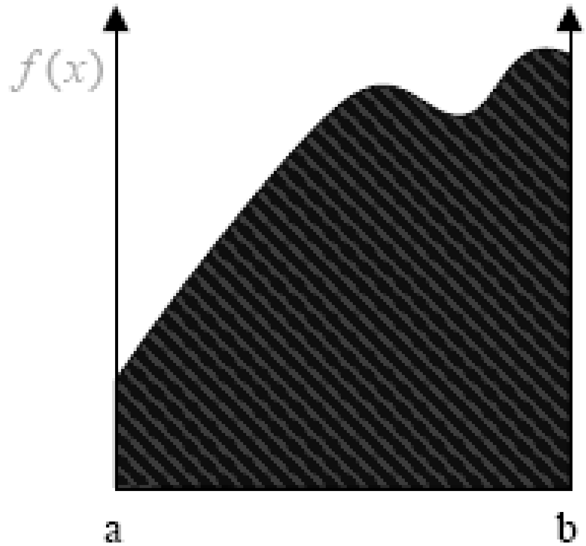
Monte Carlo Integration

- Numerical tool to evaluate integrals
  - Use sampling
- Stochastic errors
- Unbiased
  - On average, we get the right answer

Numerical Integration

- A one-dimensional integral:

$$I = \int_a^b f(x)dx$$

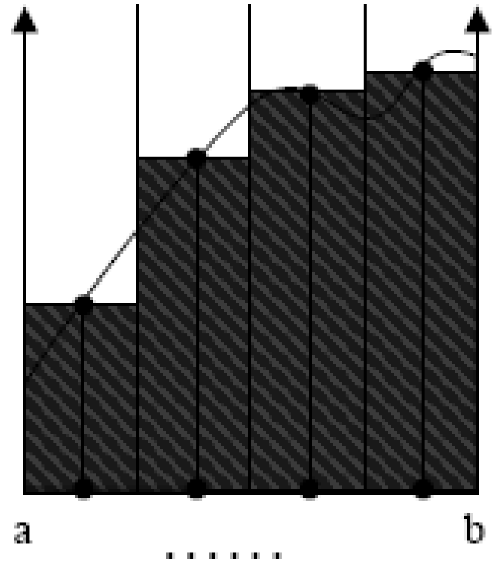


Deterministic Integration

- Quadrature rules:

$$I = \int_a^b f(x)dx$$

$$\approx \sum_{i=1}^{N} w_i f(x_i)$$

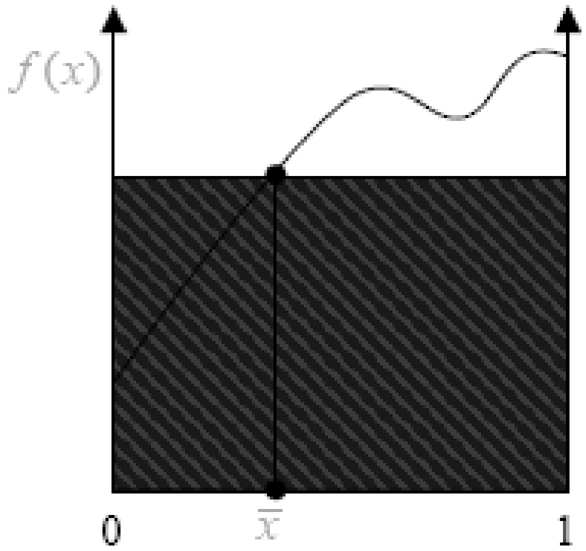


Monte Carlo Integration

Primary estimator:

$$I = \int_a^b f(x)dx$$

$$I_{prim} = f(\bar{x})$$



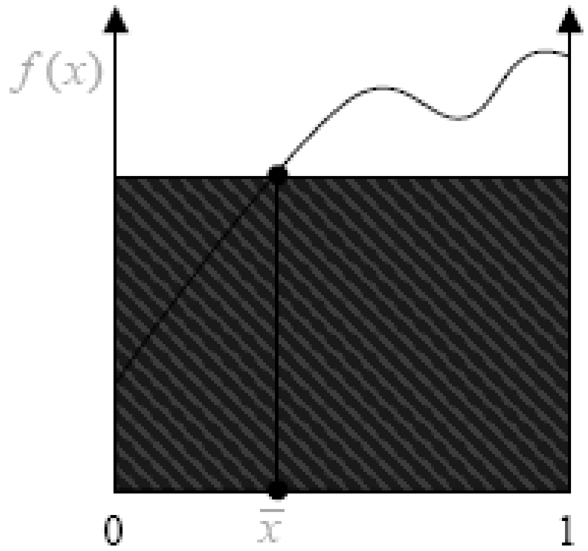
Monte Carlo Integration

Primary estimator:

$$I = \int_a^b f(x)dx$$

$$I_{prim} = f(\bar{x})$$

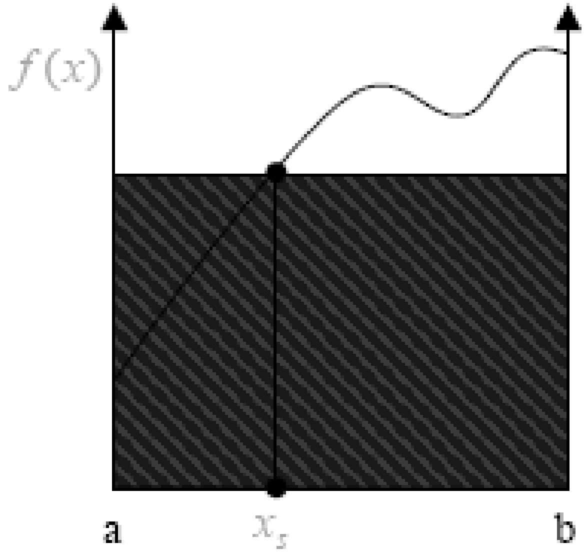
$$E(I_{prim}) = \int_0^1 f(x)p(x)dx = \int_0^1 f(x)1dx = I$$



Unbiased estimator!

Monte Carlo Integration

Primary estimator:



$$I = \int_a^b f(x)dx$$

$$I_{prim} = f(x_s)(b-a)$$

$$E(I_{prim}) = \int_a^b f(x)(b-a)p(x)dx = \int_a^b f(x)(b-a)\frac{1}{(b-a)}dx = I$$

Unbiased estimator!

Monte Carlo Integration: Error

Variance of the estimator → a measure of the stochastic error

$$\sigma^2_{prim} = \int_a^b [\frac{f(x)}{p(x)} - I]^2 \, p(x) dx$$

- **Consider p(x) for estimate**
- **We will study it as importance sampling later**

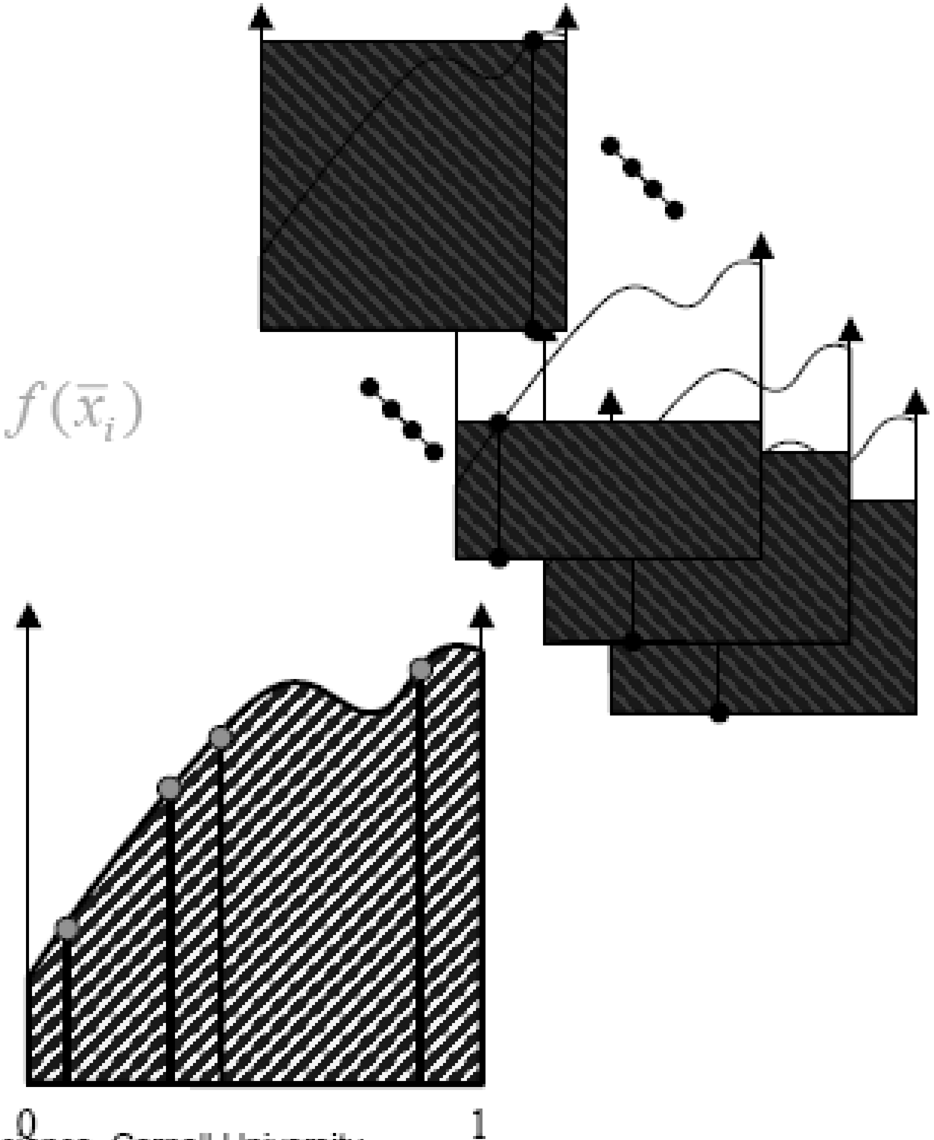
More samples

Secondary estimator

Generate N random samples x_i

Estimator: $\langle I \rangle = I_{sec} = \frac{1}{N}\sum_{i=1}^{N} f(\bar{x}_i)$

Variance $\sigma^2_{sec} = \sigma^2_{prim} / N$



Monte Carlo Integration

- Expected value of estimator

$$E[\langle I \rangle] = E[\frac{1}{N}\sum_i^N \frac{f(x_i)}{p(x_i)}] = \frac{1}{N}\int (\sum_i^N \frac{f(x_i)}{p(x_i)}) p(x)dx$$

$$= \frac{1}{N}\sum_i^N \int (\frac{f(x)}{p(x)}) p(x)dx$$

$$= \frac{N}{N}\int f(x)dx = I$$

  – on 'average' get right result: **unbiased**

- Standard deviation σ is a measure of the stochastic error

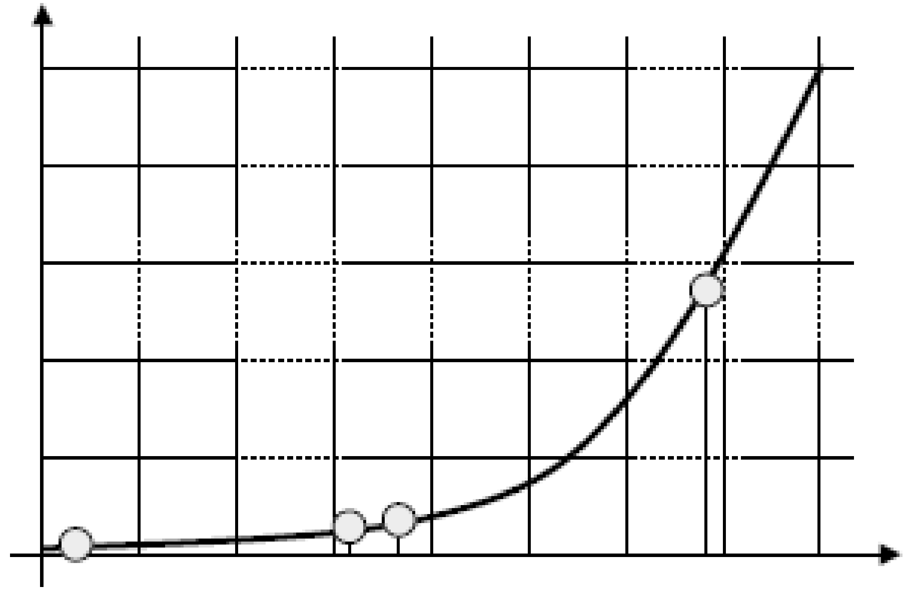
$$\sigma^2 = \frac{1}{N}\int_a^b [\frac{f(x)}{p(x)} - I]^2 p(x)dx$$

MC Integration - Example

– Integral $I = \int_0^1 5x^4 dx = 1$

– Uniform sampling

– Samples :



$x_1 =$ .86 $\langle I \rangle = 2.74$

$x_2 =$ .41 $\langle I \rangle = 1.44$

$x_3 =$ .02 $\langle I \rangle = 0.96$

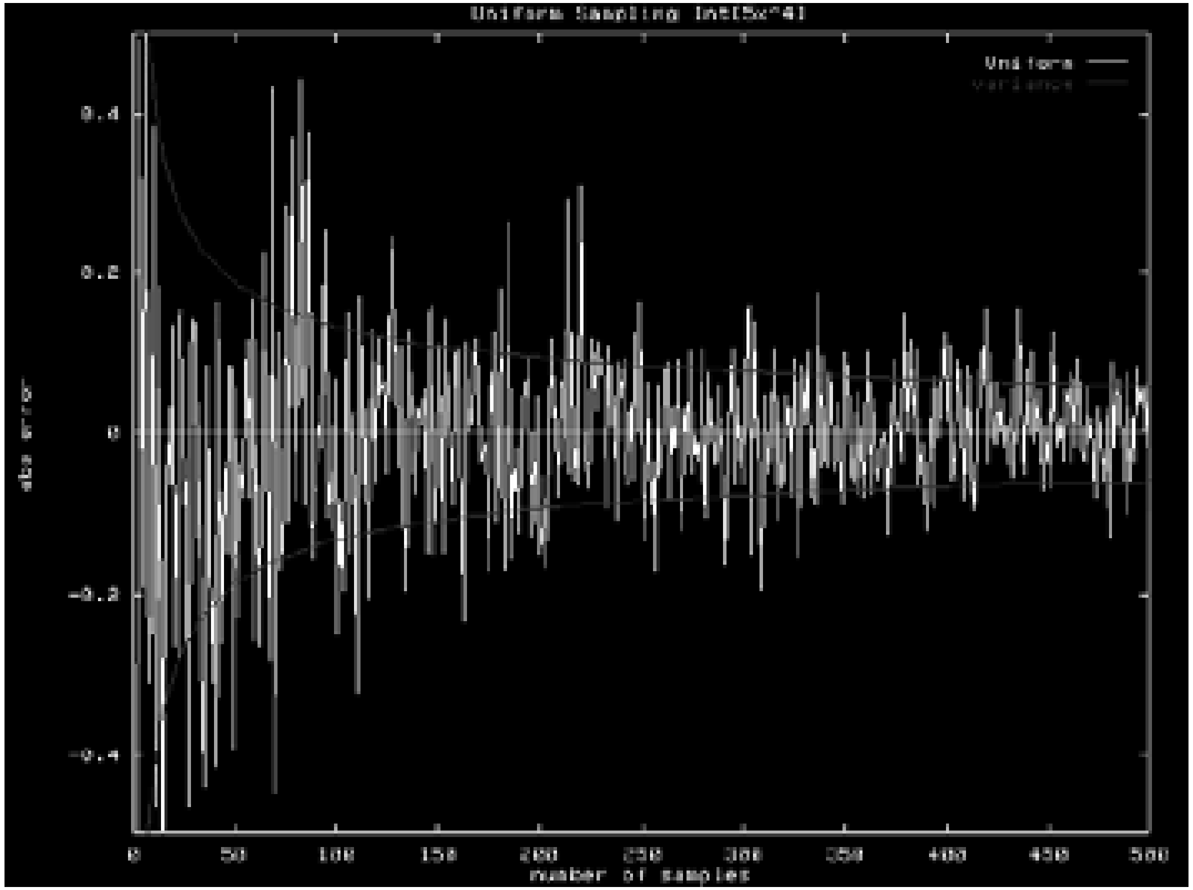
$x_4 =$ .38 $\langle I \rangle = 0.75$

MC Integration - Example

- **Integral**

$$I = \int_0^1 5x^4 dx = 1$$

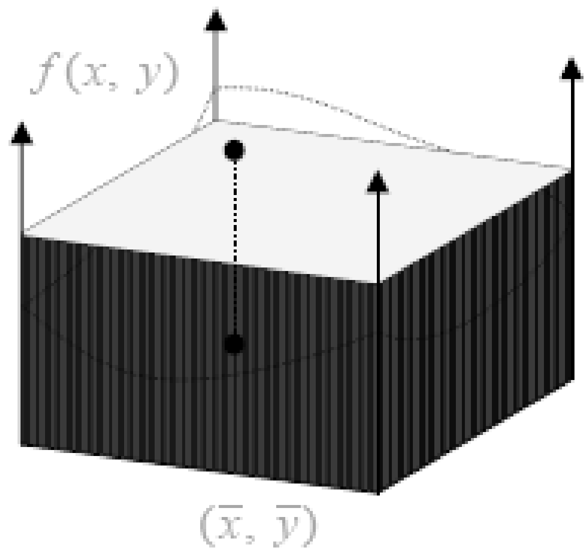
- **Variance**



MC Integration: 2D

- Primary estimator:

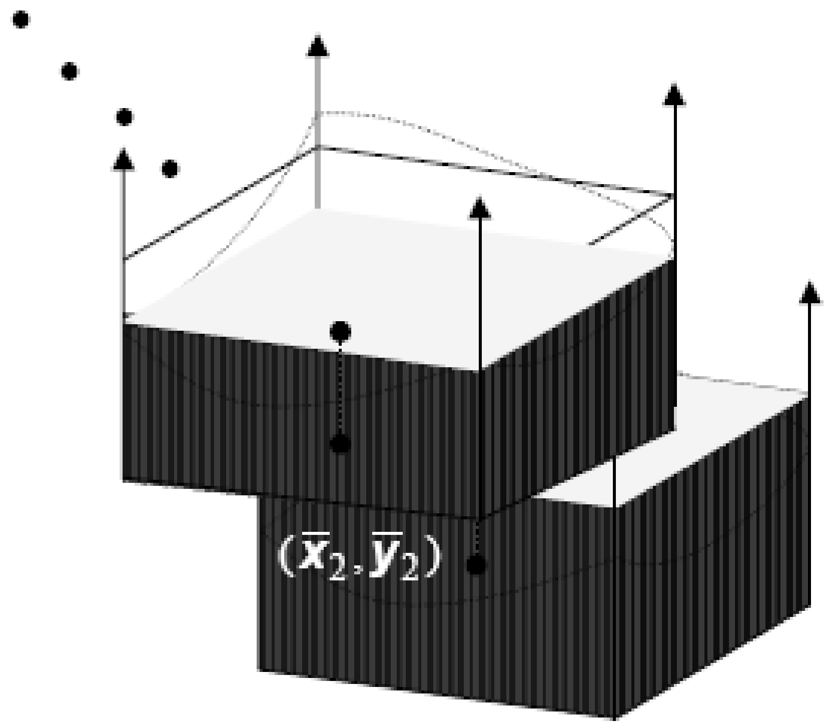
$$\bar{I}_{prim} = \frac{f(\bar{x},\bar{y})}{p(\bar{x},\bar{y})}$$



MC Integration: 2D

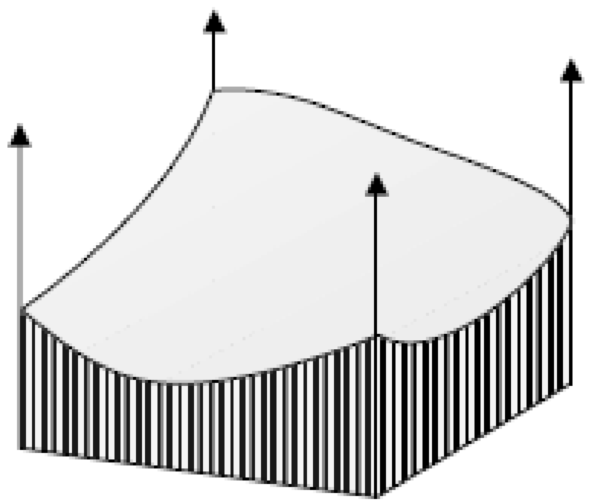
- Secondary estimator:

$$I_{sec} = \frac{1}{N}\sum_{i=1}^{N}\frac{f(\bar{x}_i, \bar{y}_i)}{p(\bar{x}_i, \bar{y}_i)}$$



Monte Carlo Integration - 2D

- MC Integration works well for higher dimensions
- Unlike quadrature



$$I = \int_a^b \int_c^d f(x, y)\,dx\,dy$$

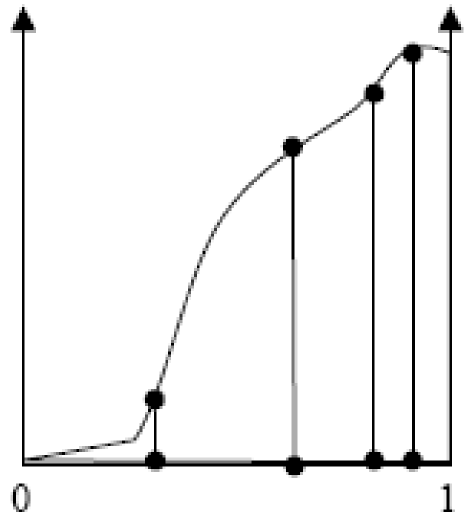
$$\langle I \rangle = \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i, y_i)}{p(x_i, y_i)}$$

Advantages of MC

- Convergence rate of $O(\frac{1}{\sqrt{N}})$
- Simple
  - Sampling
  - Point evaluation
- General
  - Works for high dimensions
  - Deals with discontinuities, crazy functions, etc.

Importance Sampling

- **Take more samples in important regions, where the function is large**



From kavita's slides

Class Objectives were:

- Sampling approach for solving the rendering equation
  - Monte Carlo integration
  - Estimator and its variance

Next Time…

- Monte Carlo ray tracing

Homework

- **Go over the next lecture slides before the class**
- **Watch 2 SIG/I3D/HPG videos and submit your summaries every Tue. class**
  - **Just one paragraph for each summary**

Example:

Title: XXX XXXX XXXX

Abstract: this video is about accelerating the performance of ray tracing. To achieve its goal, they design a new technique for reordering rays, since by doing so, they can improve the ray coherence and thus improve the overall performance.

Any Questions?

- **Submit four times in Sep./Oct.**
- **Come up with one question on what we have discussed in the class and submit at the end of the class**
  - **1 for typical questions**
  - **2 for questions that have some thoughts or surprise me**