CS680: Monte Carol Integration

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Course URL:
http://jupiter.kaist.ac.kr/~sungeui/SGA/
Course Administration

- HW
  - Due is this Thur.
Previous Time

- Radiometry
- Rendering equation
Two Forms of the Rendering Equation

- Hemisphere integration

\[ L(x \rightarrow \Theta) = L_e(x \rightarrow \Theta) + \int_{\Omega_x} f_r(\Psi \leftrightarrow \Theta) \cdot L(x \leftarrow \Psi) \cdot \cos \theta_x \cdot d\omega_\Psi \]

- Area integration

\[ L(x \rightarrow \Theta) = L_e(x \rightarrow \Theta) + \int_A f_r(\Psi \leftrightarrow \Theta) \cdot L(y \rightarrow -\Psi) \cdot \frac{\cos \theta_x \cdot \cos \theta_y}{r_{xy}^2} \cdot V(x, y) \cdot dA_y \]

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Radiance Evaluation

- Fundamental problem in GI algorithm
  - Evaluate radiance at a given surface point in a given direction
  - Invariance defines radiance everywhere else

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Radiance Evaluation

... find paths between sources and surfaces to be shaded
Why Monte Carlo?

- **Radiance is hard to evaluate**
  \[
  L(x \to \Theta) = L_s(x \to \Theta) + \int_{\Omega_x} f_r(\Psi \leftrightarrow \Theta) \cdot L(x \leftrightarrow \Psi) \cdot \cos(\Psi, n_x) \cdot d\Omega_y
  \]

- **Sample many paths**
  - Integrate over all incoming directions
- **Analytical integration is difficult**
  - Need numerical techniques
Monte Carlo Integration

- Numerical tool to evaluate integrals
- Use sampling
- Stochastic errors
- Unbiased
  - On average, we get the right answer
Probability

- Random variable $x$
- Possible outcomes: $x_1, x_2, x_3, \ldots, x_n$
  - each with probability: $p_1, p_2, p_3, \ldots, p_n$

- E.g. ‘average die’: 2, 3, 3, 4, 4, 5
  - outcomes: $x_1 = 2, x_2 = 3, x_3 = 4, x_3 = 5$
  - probabilities:
    $$p_1 = \frac{1}{6}, p_2 = \frac{1}{3}, p_3 = \frac{1}{3}, p_3 = \frac{1}{6}$$
Expected value

- Expected value = average value

\[ E[x] = \sum_{i=1}^{n} x_i p_i \]

- E.g. die:

\[ E[x] = 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{3} + 4 \cdot \frac{1}{3} + 5 \cdot \frac{1}{6} = 3.5 \]
Variance

• Expected ‘distance’ to expected value

\[ \sigma^2[x] = E[(x - E[x])^2] \]

• E.g. die:

\[ \sigma^2[x] = (2 - 3.5)^2 \cdot \frac{1}{6} + (3 - 3.5)^2 \cdot \frac{1}{3} + (4 - 3.5)^2 \cdot \frac{1}{3} + (5 - 3.5)^2 \cdot \frac{1}{6} \]

\[ = 0.916 \]

• Property: \[ \sigma^2[x] = E[x^2] - E[x]^2 \]
Continuous random variable

- Random variable \( x \in [a, b] \)
- Probability density function (pdf) \( p(x) \)
- Probability that variable has value \( x \): \( \int_a^b p(x) \, dx = 1 \)

- Cumulative distribution function (CDF)
  - CDF is non-decreasing, positive

\[ \Pr(x \leq y) = CDF(y) = \int_{-\infty}^{y} p(x) \, dx \]
Continuous random variable

• Expected value: 
  \[ E[x] = \int_a^b x p(x) \, dx \]

  \[ E[g(x)] = \int_a^b g(x) p(x) \, dx \]

• Variance:
  \[ \sigma^2[x] = \int_a^b (x - E[x])^2 p(x) \, dx \]

  \[ \sigma^2[g(x)] = \int_a^b (g(x) - E[g(x)])^2 p(x) \, dx \]

• Deviation: \( \sigma[x], \sigma[g(x)] \)
Continuous random variable

\[ P(x) \]

\[ \int_a^b p(x) \, dx = 1 \]

\[ \Pr(x \leq y) = CDF(y) = \int_{-\infty}^y p(x) \, dx \]

Probability that \( x \) belongs to \([a',b']\)

\[ = \Pr(x \leq b') - \Pr(x \leq a') \]

\[ = \int_{-\infty}^{b'} p(x) \, dx - \int_{-\infty}^{a'} p(x) \, dx = \int_{a'}^{b'} p(x) \, dx \]
Uniform distribution

\[ \int_a^b p(x) \, dx = 1 \]

\[ p(x) = \frac{1}{1-0} = 1 \]

\[ \Pr(x \in [a', b']) = \int_{a'}^{b'} 1 \, dx = (b' - a') \]

\[ \Pr(x \leq y) = CDF(y) = \int_{-\infty}^{y} p(x) \, dx = y \]

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Uniform distribution

\[ p(x) = \frac{1}{b-a} \]

\[ \int_a^b p(x) \, dx = 1 \]

Probability that \( x \) belongs to \([a',b']\) = \[ \int_{a'}^{b'} \frac{1}{b-a} \, dx = \frac{(b'-a')}{(b-a)} \]

\[ \Pr(x \leq y) = CDF(y) = \int_{-\infty}^{y} p(x) \, dx = \frac{(y-a)}{(b-a)} \]

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More than one sample

- Consider the weighted sum of N samples

- Expected value
  \[ E\left(\frac{1}{N} (x^1 + x^2 + x^3 + \ldots x^N)\right) = E[x] \]

- Variance
  \[ \sigma^2 \left[ \frac{1}{N} (x^1 + x^2 + x^3 + \ldots x^N) \right] = \frac{1}{N} \sigma^2[x] \]

- Deviation
  \[ \sigma \left[ \frac{1}{N} (x^1 + x^2 + x^3 + \ldots x^N) \right] = \frac{1}{\sqrt{N}} \sigma[x] \]
More than one sample

- Consider the weighted sum of N samples
  \[ g(x) = \frac{1}{N} \left( f(x_1) + f(x_2) + f(x_3) + \ldots + f(x_N) \right) \]

- Expected value
  \[ E[g(x)] = E \left[ \frac{1}{N} \sum_{i}^{N} f(x_i) \right] = E[f(x)] \]

- Variance
  \[ \sigma^2[g(x)] = \sigma^2 \left[ \frac{1}{N} \sum_{i}^{N} f(x_i) \right] = \frac{1}{N} \sigma^2[f(x)] \]

- Deviation
  \[ \sigma[g(x)] = \frac{1}{\sqrt{N}} \sigma[f(x)] \]
Numerical Integration

- A one-dimensional integral:

\[ I = \int_a^b f(x) \, dx \]
Deterministic Integration

- Quadrature rules:

\[ I = \int_{a}^{b} f(x) \, dx \]

\[ \approx \sum_{i=1}^{N} w_i f(x_i) \]
Monte Carlo Integration

Primary estimator:

\[ I = \int_{a}^{b} f(x) \, dx \]

\[ I_{\text{prim}} = f(\bar{x}) \]
Monte Carlo Integration

Primary estimator:

\[ I = \int_{a}^{b} f(x) \, dx \]

\[ I_{\text{prim}} = f(\bar{x}) \]

\[ E(I_{\text{prim}}) = \int_{0}^{1} f(x) \, p(x) \, dx = \int_{0}^{1} f(x) \, 1 \, dx = I \]

Unbiased estimator!

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Monte Carlo Integration

Primary estimator:

\[ I = \int_{a}^{b} f(x)\,dx \]

\[ I_{\text{prim}} = f(x_s)(b - a) \]

\[ E(I_{\text{prim}}) = \int_{a}^{b} f(x)(b - a)p(x)\,dx = \int_{a}^{b} f(x)(b - a)\frac{1}{(b - a)}\,dx = I \]

Unbiased estimator!

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Monte Carlo Integration: Error

Variance of the estimator → a measure of the stochastic error

\[ \sigma^2_{prim} = \int_a^b \left( \frac{f(x)}{p(x)} - I \right)^2 p(x) dx \]

- Consider \( p(x) \) for estimate
- We will study it as importance sampling later
More samples

Secondary estimator

Generate N random samples $x_i$

Estimator:

$$\langle I \rangle = I_{sec} = \frac{1}{N} \sum_{i=1}^{N} f(\bar{x}_i)$$

Variance

$$\sigma_{sec}^2 = \frac{\sigma_{prim}^2}{N}$$
More samples

Secondary estimator

Generate $N$ random samples $x_i$

Estimator: $\langle I \rangle = I_{sec} = \frac{1}{N} \sum_{i=1}^{N} f(x^i)(b-a)$

Variance $\sigma^2_{sec} = \sigma^2_{prim} / N$

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Monte Carlo Integration

• Expected value of estimator

\[ E[\langle I \rangle] = E\left[ \frac{1}{N} \sum_i^N \frac{f(x_i)}{p(x_i)} \right] = \frac{1}{N} \int \left( \sum_i^N \frac{f(x_i)}{p(x_i)} \right) p(x) dx \]

\[ = \frac{1}{N} \sum_i^N \int \left( \frac{f(x)}{p(x)} \right) p(x) dx \]

\[ = \frac{N}{N} \int f(x) dx = I \]

– on ‘average’ get right result: unbiased

• Standard deviation \( \sigma \) is a measure of the stochastic error

\[ \sigma^2 = \frac{1}{N} \int_a^b \left[ \frac{f(x)}{p(x)} - I \right]^2 p(x) dx \]
MC Integration - Example

- Integral
  \[ I = \int_{0}^{1} 5x^4 \, dx = 1 \]

- Uniform sampling

- Samples:

  \[ x_1 = 0.86 \quad <I> = 2.74 \]
  \[ x_2 = 0.41 \quad <I> = 1.44 \]
  \[ x_3 = 0.02 \quad <I> = 0.96 \]
  \[ x_4 = 0.38 \quad <I> = 0.75 \]
MC Integration - Example

- Integral

\[ I = \int_{0}^{1} 5x^4 \, dx = 1 \]

- Variance
MC Integration: 2D

- Primary estimator:

\[
\bar{I}_{\text{prim}} = \frac{f(\bar{x}, \bar{y})}{p(\bar{x}, \bar{y})}
\]
MC Integration: 2D

- Secondary estimator:

\[ I_{\text{sec}} = \frac{1}{N} \sum_{i=1}^{N} \frac{f(\bar{x}_i, \bar{y}_i)}{p(\bar{x}_i, \bar{y}_i)} \]
Monte Carlo Integration - 2D

- MC Integration works well for higher dimensions
- Unlike quadrature

\[ I = \int_a^b \int_c^d f(x, y) \, dx \, dy \]

\[ \langle I \rangle = \frac{1}{N} \sum_{i=1}^N \frac{f(x_i, y_i)}{p(x_i, y_i)} \]
Advantages of MC

- **Convergence rate of** $O\left(\frac{1}{\sqrt{N}}\right)$

- **Simple**
  - Sampling
  - Point evaluation

- **General**
  - Works for high dimensions
  - Deals with discontinuities, crazy functions, etc.
MC Integration - 2D example

- Integration over hemisphere:

\[
I = \int_{\Omega} f(\Theta) d\omega_{\Theta} = \int_{0}^{2\pi} \int_{0}^{\pi/2} f(\varphi, \theta) \sin \theta d\theta d\varphi
\]

\[
\langle I \rangle = \frac{1}{N} \sum_{i=1}^{N} \frac{f(\varphi_i, \theta_i) \sin \theta}{p(\varphi_i, \theta_i)}
\]
Hemisphere Integration example

Irradiance due to light source:

\[
I = \int_{\Omega} L_{\text{source}} \cos \theta d\omega \\
= \int_{0}^{\pi/2} \int_{0}^{2\pi} L_{\text{source}} \cos \theta \sin \theta d\theta d\phi
\]

\[
p(\omega_i) = \frac{\cos \theta \sin \theta}{\pi}
\]

\[
\langle I \rangle = \frac{1}{N} \sum_{i=1}^{N} \frac{L_{\text{source}}(\omega_i) \cos \theta \sin \theta}{p(\omega_i)} = \frac{\pi}{N} \sum_{i=1}^{N} L_{\text{source}}(\omega_i)
\]

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Importance Sampling

- Take more samples in important regions, where the function is large.

From kavita’s slides
MC integration - Non-Uniform

- Some parts of the integration domain have higher importance
- Generate samples according to density function \( p(x) \)

\[
\langle I \rangle = \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i)}{p(x_i)}
\]

- Estimator?

- What is optimal \( p(x) \)?

\[ p(x) \approx f(x) / \int f(x) \, dx \]
MC integration - Non-Uniform

• Generate samples according to density function $p(x)$

$$p(x) \approx \frac{f(x)}{\int f(x) \, dx}$$

• Why?

$$I_{\text{estimator}} = \frac{1}{N} \sum \frac{f(x)}{p(x)} = \frac{1}{N} \sum \frac{f(x)}{f(x) / I} = \frac{1}{N} \sum I = I$$

$$\sigma^2 = \frac{1}{N} \int_a^b \left[ \frac{f(x)}{p(x)} - I \right]^2 p(x) \, dx$$

• But.....

$$= \frac{1}{N} \int_a^b \left[ \frac{f(x)}{f(x) / I} - I \right]^2 p(x) \, dx = 0$$
Example

- Function: \[ I = \int_{0}^{4} x \, dx = 8 \quad \text{and} \quad f(x) = x \]

\[
\sigma^2 = \frac{1}{N} \int_{a}^{b} \left[ \frac{f(x)}{p(x)} - I \right]^2 p(x) \, dx
\]

\[
p(x) = \frac{x}{8}, \quad \sigma^2 = 0 \quad I_{\text{estimator}} = I = 8
\]

\[
p(x) = \frac{1}{4}, \quad \sigma^2 = \frac{1}{N} \int_{0}^{4} \left[ \frac{x}{1/4} - 8 \right]^2 \frac{1}{4} \, dx = 21.3 / N
\]

\[
p(x) = \frac{x+2}{16}, \quad \sigma^2 = \frac{1}{N} \int_{0}^{4} \left[ \frac{x}{(x+2)/16} - 8 \right]^2 \frac{x+2}{16} \, dx = 6.3 / N
\]

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Importance Sampling

• Generate samples from density function \( p(x) \)

\[
\langle I \rangle = \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i)}{p(x_i)}
\]

• Optimal \( p(x) \)? \( p(x) \approx \frac{f(x)}{\int f(x)dx} \)

• General principle:
  – Closer shape of \( p(x) \) is to shape of \( f(x) \), lower the variance

• Variance can increase if \( p(x) \) is chosen badly
Sampling according to pdf

- Inverse cumulative distribution function
- Rejection sampling
Inverse Cumulative Distribution Function – Discrete Case

- Consider discrete events $x_i$
  - with probability $p_i$

- Select $x_i$ if:
  $$ p_1 + \ldots + p_{i-1} < \xi < p_1 + \ldots + p_{i-1} + p_i $$

$$ \sum_{j=1}^{i-1} p_j < \xi < \sum_{j=1}^{i} p_j $$

$$ P(x_i) = P(\xi \in [\sum_{j=1}^{i-1} p_j, \sum_{j=1}^{i} p_j]) $$

$$ P(a < \xi < b) = (b - a) $$

$$ P(x_i) = P(\xi \in [\sum_{j=1}^{i-1} p_j, \sum_{j=1}^{i} p_j]) = \sum_{j=1}^{i} p_j - \sum_{j=1}^{i-1} p_j = p_i $$

, given uniform sampling

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Continuous Random Variable

- **Algorithm**
  - Pick $u$ uniformly from $[0, 1)$
  - Output $y = P^{-1}(u)$, where $P(y) = \int_{-\infty}^{y} p(x)dx$
1) Choose a normalized probability density function $p(x)$
Non-Uniform Samples

1) Choose a normalized probability density function \( p(x) \)

2) Integrate to get a cumulative probability distribution function \( P(x) \):

\[
P(x) = \int_{0}^{x} p(t)dt
\]

Note this is similar to computing \( \sum_{j=1}^{i} p_j \)

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Non-Uniform Samples

1) Choose a normalized probability density function $p(x)$

2) Integrate to get a probability distribution function $P(x)$:

$$P(x) = \int_{0}^{x} p(t)dt$$

3) Invert $P$:

$$x = P^{-1}(\xi)$$

Note this is similar to going from y axis to x in discrete case!

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Cosine distribution

\[ f = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} \cos \theta \sin \theta \, d\theta \, d\phi \]

\[ p(\theta, \phi) = \frac{\cos \theta \sin \theta}{\pi} \]

\[ CDF(\theta, \phi) = \int_0^\theta \int_0^\phi \frac{\cos \theta \sin \theta}{\pi} \, d\theta \, d\phi = (1 - \cos^2 \theta) \frac{\phi}{2\pi} \]

\[ F(\theta) = 1 - \cos^2 \theta \]

\[ F(\phi) = \frac{\phi}{2\pi} \]

\[ \phi_i = 2\pi \xi_1 \quad \theta_i = \cos^{-1} \sqrt{\xi_2} \]
Rejection Method

- Often not possible to compute the inverse of cdf
  - Pick $\xi_1, \xi_2$

$$I = \int_{a}^{b} f(x) \, dx$$

- If $\xi_2 < f(\xi_1)$, select $\xi_1$

- Is this efficient? What determines efficiency? $\frac{A(f)}{A(\text{rectangle})}$
Summary

- Monte Carlo integration
- Estimators
- Sampling non-uniform distribution
Next Time

- Monte Carlo ray tracing